Bifurcation of synchronous oscillations into torus in a system of two reciprocally inhibitory silicon neurons: Experimental observation and modeling

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Oscillatory activity in the central nervous system is associated with various functions, like motor control, memory formation, binding, and attention. Quasiperiodic oscillations are rarely discussed in the neurophysiological literature yet they may play a role in the nervous system both during normal function and disease. Here we use a physical system and a model to explore scenarios for how quasiperiodic oscillations might arise in neuronal networks. An oscillatory system of two mutually inhibitory neuronal units is a ubiquitous network module found in nervous systems and is called a half-center oscillator. Previously we created a half-center oscillator of two identical oscillatory silicon (analog Very Large Scale Integration) neurons and developed a mathematical model describing its dynamics. In the mathematical model, we have shown that an in-phase limit cycle becomes unstable through a subcritical torus bifurcation. However, the existence of this torus bifurcation in experimental silicon two-neuron system was not rigorously demonstrated or investigated. Here we demonstrate the torus predicted by the model for the silicon implementation of a half-center oscillator using complex time series analysis, including bifurcation diagrams, mapping techniques, correlation functions, amplitude spectra, and correlation dimensions, and we investigate how the properties of the quasiperiodic oscillations depend on the strengths of coupling between the silicon neurons. The potential advantages and disadvantages of quasiperiodic oscillations (torus) for biological neural systems and artificial neural networks are discussed. © 2004 American Institute of Physics. [DOI: 10.1063/1.1795471]

Investigation of mechanisms of rhythms generation in the brain is an important, but still unresolved task in modern neuroscience. On the one hand, the brain of mammals contains a huge number of neurons that possess different structures and properties. On the other hand, it operates as a whole organ, with ordered dynamics of all neurons. The order in neural systems is established through neuronal interaction, their synchronization, and rhythm generation. Neuronal networks in invertebrates contain considerably fewer neurons thus allowing for detailed investigations of mechanisms of neuronal interactions and oscillation generation. In this article, we study a mechanism for the generation of quasiperiodic oscillations in a system of two mutually inhibitory silicon neurons and their mathematical model. A similar mechanism for generation of quasiperiodic oscillations can potentially apply to the central pattern generator of the lamprey swim system and the leech heartbeat system, which are based on half-center oscillators. We found that our two-neuron system demonstrates quasiperiodic oscillations (torus), along with typical antiphase or in-phase oscillations. The existence of quasiperiodic oscillations was demonstrated by complex time series analysis, including bifurcation diagrams, mapping techniques, correlation functions, amplitude spectra, and correlation dimensions. Experimental data compare well with the simulation results obtained from the mathematical model of the two-neuron system.

I. INTRODUCTION

Nonlinear oscillations appear in physical, chemical, and biological systems. Oscillatory electrical activity is widely observed in nervous systems and has been linked to...
such functions as motor control, memory formation, binding, novelty detection, attention, and information processing. In classical behavioral studies by von Holst (1939, 1973), two phenomena were described—absolute and relative coordination. Oscillatory activity in neurons is commonly represented by limit cycles in mathematical models and synchronization of their activity through entrainment or mutual interactions represent absolute coordination, or magnet effect in von Holst’s terminogy.

Experimental data from cardiology and electroencephalography suggest that periodic oscillations, like those described by limit cycles in models, indicate a pathological state of a living system. It has been suggested that such oscillations are too “rigid” and difficult to control. We suggest that quasiperiodic oscillations may provide advantages for control in living systems and may be of particular importance for the control of rhythmic movements. They are a realization of relative coordination phenomenon described by von Holst. Quasiperiodic oscillations can produce on average appropriate phase relationship but yet be more “flexible” and amenable to control compared to limit cycle oscillations. Theoretical work indicates that quasiperiodic oscillations can increase information flow and improve neural network performance, compared to periodic oscillations with a single frequency.

The genesis, dynamics, and properties of quasiperiodic oscillations (torus), have been studied in several early and recent publications. To identify biophysically plausible mechanisms for the generation of quasiperiodic oscillations that are sufficiently robust to persist under physical conditions is thus of considerable interest. One possible mechanism is a torus bifurcation in a system of symmetrically connected identical oscillators. It is a generic bifurcation describing a scenario in which stable phase-locked oscillations can lose stability giving rise to stable quasiperiodic oscillations. Quasiperiodic oscillations may be anticipated in behavioral and neurophysiological experiments not only as a regime of normal operation but also as a pattern indicating oncoming state transition. Such torus bifurcations have been observed in mathematical models of either electrotonically or synaptically coupled neuronal oscillators. It is crucial to test robustness of this mechanism in a physical experiment to demonstrate that quasiperiodic oscillations may be functionally employed by living systems. Because living neurons are generally sensitive to experimental treatment and their state decays during the experiment, such testing may be done more easily with silicon [analog Very Large Scale Integration (aVLSI)] neurons that are physical implementations which have dynamics similar to those of a living neuron. This approach allows one to directly apply rigorous methods developed in dynamical system theory.

In this article, we study the appearance of quasiperiodic oscillations, i.e., oscillations with noncommensurate frequencies, in a system of two identical aVLSI neurons with mutually inhibitory connections. Such neural subsystems with mutually inhibitory coupling, called half-center oscillators, are ubiquitous in nervous systems. For example, they are crucial for robust rhythm generation in central pattern generators of the lamprey swim system, heartbeat in leeches, and many invertebrate systems. Depending on the neuronal connection strength, living half-center oscillators can show different types of activity and synchronization. For example, reduction of the strength of inhibitory connections by strychnine in lamprey swim system leads to synchronous oscillations. Quasiperiodic oscillations in living half-center oscillator have not yet been observed, but here we demonstrate with mathematical and physical models that this common neuronal structure can produce such oscillations.

Previously, we have developed and studied the dynamics of an aVLSI neuron inspired by Morris–Lecar oscillator. We have also developed a mathematical model of this silicon neuron that accurately describes its experimentally observed dynamics. Here we use a pair of such silicon neurons to create and study a half-center oscillator. We have shown that this physical system and its mathematical model can demonstrate a number of different phase-locked oscillations depending on the strength of synaptic coupling. Experimentally, we investigated qualitative behavior of the system with different parameters of the silicon neurons. Mismatch in the inherent frequencies of the two coupled oscillators would dominantly support asymmetric quasiperiodic oscillations. Quasiperiodic oscillations are commonly discussed in the context of such mismatched frequencies of coupled oscillators, or mismatched frequencies of entraining signal and the native frequency of an oscillator. We focused our study on the symmetrical case of two identical oscillators, which has been less studied experimentally and appears to be not intuitive. Silicon neurons being physical objects are very attractive for such experimental studies in this area. Due to variations in the fabrication process, our silicon neurons are not identical oscillators. By tweaking available tuning parameters we were always able to bring frequencies of two oscillators to be very close with a variety of different parameter sets, however. Then, for each of these different parameter tunings, we observed the resultant oscillatory modes (in-phase, phase-shifted, antiphase, or drifting) for weak coupling and further worked with those tunings that supported synchronous oscillations. With further parameter tweaking, we then matched the form of the oscillations relatively well for different strengths of coupling. Then as the strength of coupling was varied, we could observe transitions between different oscillatory regimes. Although some parameter sets would support only alternating oscillations, it was not difficult to find different parameter sets, which produced synchronous oscillations when coupling is weak. Experiments were done with three pairs of aVLSI neurons. All of them showed similar behavior and demonstrated bifurcation of periodic oscillation into a two-dimensional torus. For these sets, a generic situation was that strong coupling supported antiphase oscillations, weak coupling supported in-phase oscillations, and moderate coupling supported in-phase oscillations coexisting with either phase-shifted or antiphase oscillations.

Generically, besides phase-locked oscillations, we observed oscillations tentatively identified as quasiperiodic. In the mathematical model, we qualitatively reproduced all these observed phenomena. Stability of limit cycles repre-
senting phase-locked oscillations was numerically calculated using the bifurcation analysis software LocBif.\textsuperscript{36} LocBif calculates multipliers of a limit cycle along with variation of the bifurcation parameter. In the mathematical model, we have shown that the in-phase limit cycle becomes unstable through a torus bifurcation (Neimark–Sacker–Andronov–Hopf torus bifurcation) as the coupling strength reaches the bifurcation value at which a pair of complex-conjugate multipliers crossed the unit circle, but we did not rigorously demonstrate that we had observed a torus in our experiments, and did not explore torus properties either in the experiments or the model. Here we study the appearance of quasiperiodic oscillations in this experimental system, using complex time series and bifurcation analysis which include calculation of phase shift between the neurons, correlation functions, amplitude spectra, and correlation dimension. We demonstrate the existence of the torus in the experimental system as predicted by the mathematical model\textsuperscript{13} and investigate its properties at different connection strengths. Agreement between the mathematical model and experimental observations indicates the possibility that similar oscillatory regimes can be revealed in living half-center oscillators and creates a background for the development of more complex multicellular neural network models. The potential functional roles of quasiperiodic oscillations (torus) in biological neural systems and artificial neural networks are discussed.

A preliminary report of these experimental findings has appeared in conference proceedings.\textsuperscript{37}

\section*{II. MODEL}

A model of a silicon neuron has two state variables $V$ and $W$ which represent neuron’s membrane potential and activation of $K^+$ current, correspondingly. The variable $W$ has slow dynamics compared to $V$. Mutually inhibitory connections are utilized through inhibitory synapses with instantaneous dynamics. A model of two neurons with reciprocal inhibitory connections is described by set of differential equations:\textsuperscript{34}

\begin{equation}
C_1 \dot{V}_1 = \alpha_{PI} I_{ext} + \alpha_{PI}I_{BH} \frac{\exp[(\kappa(V_1 - V_H)/U_T)]}{1 + \exp[(\kappa(V_1 - V_H)/U_T)]} - \alpha_{NI} I_{BH} \frac{\exp[(\kappa(W_1 - V_I)/U_T)]}{1 + \exp[(\kappa(W_1 - V_I)/U_T)]} - \alpha_{NI} I_{HSyn} \frac{\exp[(\kappa(V_2 - V_{inh})/U_T)]}{1 + \exp[(\kappa(V_2 - V_{inh})/U_T)]},
\end{equation}

\begin{equation}
C_2 \dot{W}_1 = \beta_{PI} I_{I} \tanh(\kappa(V_1 - W_1)/2U_T),
\end{equation}

FIG. 1. Experimentally obtained (a) and theoretical (b) bifurcation diagrams for the two mutually inhibitory silicon neurons. The phase difference $\phi$ of different phase-locked oscillations is plotted against strength of the synaptic connections. Model parameters used in this and subsequent figures are: $I_a = 2.81$ nA, $I_{ext} = 15.0$ nA, $I_{BH} = 48.0$ nA, $I_{HSyn} = 6.43682$ nA, $\kappa = 0.65$, $V_{M} = 5.0$ V, $V_{inh} = 0.0$ V, $V_I = 2.0$ V, $C_1 = C_2 = 35$ pF, $U_T = 0.025$ V. (a) The bifurcation diagram of phase-locked oscillations obtained for the silicon half-center oscillator as the strength of connection $I_{HSyn}$ is varied. Synchronous oscillations are stable for weak coupling and give rise to stable drifting oscillations at the moderate strength of coupling $a$ ($I_{HSyn} = 2.7$ nA). A pair of phase-shifted oscillations appear at the point $b$ ($I_{HSyn} = 1.5$ nA) are stable for moderate strengths of coupling and give stability to alternating oscillations, merging at the point $c$ ($I_{HSyn} = 6.3$ nA). Alternating oscillations remain stable for the strong inhibitory connections. (b) The bifurcation diagram of phase-locked oscillations obtained for the mathematical half-center oscillator model as the strength of connection $I_{HSyn}$ is varied. The synchronous limit cycle, stable with weak coupling, became unstable at a subcritical Neimark–Sacker–Andronov–Hopf torus bifurcation at moderate coupling strength at the point $a$ ($I_{HSyn} = 0.140$ nA). The pair of stable nonsymmetrical phase-shifted limit cycles appears along with the pair of unstable nonsymmetrical phase-shifted limit cycles at a fold bifurcation for limit cycles at the point $b$ ($I_{HSyn} = 0.056$ nA). The antiphase limit cycle is stable for strong inhibitory connections and looses stability on the pitch-fork limit cycle bifurcation giving stability to the pair of phase-shifted limit cycles at the point $c$ ($I_{HSyn} = 6.5$ nA). Circles, triangles, and crosses represent synchronous oscillations, phase-shifted oscillations and alternating oscillations, respectively. Vertical lines mark values of synaptic strengths used in Figs. 3–6.
Here, \( V_1 \) and \( V_2 \) are the membrane potential for the first and second neurons, respectively; \( W_1 \) and \( W_2 \) are the slow state variables corresponding to activation of \( K^+ \) currents; \( C_1 \) and \( C_2 \) are the capacitors; \( I_{ext} \) is an external current which can be used as a control parameter; \( I_{BL} \) is the amplitude of output currents of the differential pair circuits; \( I_{BSyn} \) is the amplitude of synaptic current; \( I_t \) is the current scaling the time constants of \( W_1 \) and \( W_2 \). Functions \( \alpha_p, \alpha_N, \beta_p, \) and \( \beta_N \) represent ohmic effects of transistors inside the silicon neuron \((i=1,2\) for the first and second neurons, respectively), \( U_T \) is the thermal voltage; \( V_{th}, V_L, \) and \( V_{th} \) are potentials in characteristic points inside the silicon neuron. Functions \( \alpha_p, \alpha_N, \beta_p, \) and \( \beta_N \) where \( i=1,2 \) have the following forms:

\[
\alpha_p = 1 - \exp[(V_i - V_{high})/U_T],
\]

\[
\alpha_N = 1 - \exp[(V_{low} - V_i)/U_T],
\]

\[
\beta_p = 1 - \exp[(W_i - V_{ab})/U_T],
\]

\[
\beta_N = 1 - \exp(-W_i/U_T),
\]

where \( V_{high}, V_{low}, \) and \( V_{ab} \) are reference potentials in the silicon neuron defined in Refs. 33 and 34. The terms \( \alpha_N \) and \( \alpha_p \) limit the range of \( V_i \) to \( V_{low} \) and \( V_{high} \); and the terms \( \beta_N \) and \( \beta_p \) limit the range of \( W_i \) to within the supply rails (\( G_{ext} \) and \( V_{ab} \)). More detailed description of the model is given in Refs. 33 and 34.

The system of Eqs. (1)-(8) have four variables, \( V_1, V_2, W_1, \) and \( W_2 \), that are functions of time only. All other values are parameters, independent of time. In Eqs. (1) and (3), first terms represent external currents, second terms are the instantaneously activated \( \text{Ca}^{2+} \) currents, third terms are the slowly activated \( K^+ \) currents, and fourth terms are the inhibitory synaptic currents for the first and second silicon neurons, respectively. Our model is based on Morris–Lecar model of an excitable cell\(^{35}\) and represents a reasonable level of reduction of the neuronal dynamics reconciling relative simplicity of analysis with biophysical plausibility.\(^{14,38}\) In addition, this model successfully explained experimental data on appearance of synchronous oscillations in the lamprey swim system during exposure to strychnine that reduce inhibitory interneuron coupling.\(^{28,29}\)

### III. METHODS OF ANALYSIS

Model Eqs. (1)-(8) are solved by the fourth-order Runge–Kutta method with the time step 0.1 ms. The frequency spectra are calculated using the digital fast Fourier transform from both experimental and calculated time series with the number of points from \( N=2048 \) to \( N=65536.\(^{39}\) The correlation dimension \( D_2 \) (Ref. 40) is calculated, using time series with \( N=2048 \) to 16 384. In our data analysis, the value of \( D_2 \) does not exceed 2.

Correlation functions \( r_{ij}, V_{ij} \) is defined by equation

\[
r_{ij} = \frac{S_j(\tau)}{A_{ij}},
\]

where \( S_j(\tau) = \Sigma_{k=1}^{N} (V_{ki} - \overline{V}_i)(V_{kj} - \overline{V}_j) \) for \( i \) and \( j \)th neuronal outputs \((i,j=1,2)\). \( A = 6000 \) is the normalization constant. Here, \( \overline{V}_i \) and \( \overline{V}_j \) are the average values of \( V_i \) and \( V_j \), respectively.

### IV. RESULTS

Simulated bifurcation diagram for oscillation period \( T \) of neuronal activity as a function of logarithm of synaptic current is shown in Fig. 2(a). We consider only \( I_{BSyn} \) as a bifurcation parameter in a wide range from 0.02 to 100 nA. All transitions from one type of activity to another in a two-neuron system were observed only within this region. The other parameters were used for adjustment of each silicon neuron separately to generate identical periodic oscillations. Figure 2(b) shows details of bifurcation diagram in the range of \( \log(I_{BSyn}) \) from -2 to 0. Bifurcation diagram contains two branches. One branch shown in Fig. 2 by triangles corresponds to the antiphase \((I_{BSyn} \approx 6.5 \text{ nA})\) and phase shifted oscillations \((I_{BSyn} \approx 6.5 \text{ nA})\) in model two-neuron system. Below this curve (points), we observe bifurcation of synchronous oscillations to activity on a torus at the increase of \( I_{BSyn} \) from 0.02 to 0.37 nA. Torus appeared at \( I_{BSyn} = 0.11 \text{ nA} \) and exist until \( I_{BSyn} = 0.37 \text{ nA} \). Further increase in \( I_{BSyn} \) beyond 0.37 nA produces antiphase oscillations. In this region, the lower branch of the bifurcation diagram coincides with the upper branch [Fig. 2(b)].

To demonstrate the existence of a bifurcation of synchronous oscillations to a torus in our two-neuron model, we calculated maps of oscillation amplitudes \( V_1 \) versus \( V_2 \), phase shift between oscillation \( \phi \), correlation functions \( r_{ij} \), amplitude spectra, and correlation dimensions \( D_2 \) at each point of bifurcation diagram. The bifurcation was detected by the following transitions when \( I_{BSyn} \) increased from 0.10 to 0.11 nA: (1) maps of oscillation amplitudes \( V_1 \) versus \( V_2 \) changed from single line to more complex trajectory; (2) phase shift between oscillation \( \phi \) changed from straight line...

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**FIG. 2.** Bifurcation diagram of the oscillation period \( T \) when \( I_{BSyn} \) is varied. (b) shows details of bifurcation diagram of (a). Upper branch with antiphase and phase shifted oscillations is shown by triangles, lower branch with synchronous oscillations and tori is shown by points.
\( \phi = 0 \) to periodic oscillation; (3) correlation function \( r_{V_1, V_2} \), transferred from pure sinusoid to modulated sinusoid; (4) amplitude spectra changed from identical peaks with nonlinear harmonics to those with satellite frequencies; and (5) correlation dimension increased from 1 to 2.

These functions at several characteristic points of bifurcation diagram from Fig. 2 are shown in Figs. 3 and 4.

At \( I_{BSyn} = 0.02 \) nA we observe synchronous oscillations. The activities of both neurons coincide in Fig. 3(a), map of \( V_1 \) versus \( V_2 \) represents a straight segment [Fig. 3(e)], and phase shift \( \phi = 0 \). In Fig. 4(a) correlation function is a simple sinusoid. Amplitude spectra with two peaks coincide for both oscillation time series [Fig. 4(e)]. The first peak is at the main oscillation frequency 78 Hz, the second represents its nonlinear harmonics at 156 Hz. Correlation dimension \( D_2 \) is equal to 1 [Fig. 4(i)].

Increase of \( I_{BSyn} \) to 0.14 nA produces oscillations on a torus. It is seen from Figs. 3(f) and 3(j), where a map of \( V_1 \) versus \( V_2 \) demonstrates quasiperiodicity that covers figure plane and phase shift \( \phi \) shows harmonic oscillations. Additional relatively small peaks appear at the frequency spectra of both oscillations. Two of them are on either side of the main peak, another two are near its harmonics [Fig. 4(f)]. Correlation dimension \( D_2 \) is equal to 2 at small values of \( \ln \varepsilon \), showing quasiperiodicity, and is equal to 1 at larger \( \ln \varepsilon \), because of the contribution of the second frequency is very small at larger \( \ln \varepsilon \).

The quasiperiodic oscillations (torus) with larger amplitude of the second frequency is observed at \( I_{BSyn} = 0.23 \) nA. The map of \( V_1 \) versus \( V_2 \) cover a larger area of plot in Fig. 3(g), phase shift has harmonic oscillations with larger amplitudes [Fig. 3(k)]. The correlation function, \( r_{V_1, V_2} \), is modulated significantly by the difference between the two frequencies. Frequency spectra show multiple amplitudes around the main oscillation frequency and its harmonics [Fig. 4(g)]. We explain multiple peaks by stronger nonlinearity of oscillations in comparison with Fig. 4(f). Correlation dimension \( D_2 \) is equal to 2 in a wide range of \( \ln \varepsilon \).

Finally, at the relatively large synaptic current \( I_{BSyn} = 75.0 \) nA, the oscillations of the two neurons become strongly nonlinear and antiphase. It is clearly seen from simulated time series of neuron amplitudes [Fig. 3(d)]. Phase shift \( \phi \) is constant and equal to 0.5 [Fig. 3(i)], correlation function \( r_{V_1, V_2} \), has a minimum value at \( \tau = 0 \) [Fig. 4(d)]. Multiple equidistant peaks appear on the oscillation spectra, showing strong nonlinearity of oscillations [Fig. 4(h)]. The correlation dimension \( D_2 = 1 \).

A similar scenario of bifurcation of the synchronous oscillations into a torus is observed experimentally with aVLSI neurons. Figures 5 and 6 show maps of \( V_1 \) versus \( V_2 \) [Figs. 5(e)–5(h)], phase between outputs of two neurons [Figs. 5(i)–5(l)], correlation function for two neuron outputs [Figs. 6(a)–6(d)], amplitude spectra [Figs. 6(e)–6(h)], and correlation dimensions \( D_2 \) [Figs. 6(i)–6(l)] calculated from experimental time series.

Below \( I_{BSyn} = 2.7 \) nA, synchronous oscillations were observed. An example of this type of activity at \( I_{BSyn} = 2.6 \) nA is shown in Figs. 5 and 6. In this case, the correlation function is a periodic function [Fig. 6(a)], amplitude spectra [Fig. 6(e)] show only one component at 69.2 Hz with a second harmonic at 138.0 Hz, and correlation dimension \( D_2 = 1 \).

With an increase of \( I_{BSyn} \) to 2.7 nA, quasiperiodic activity (torus) appears with two frequencies, 54.8 and 70.2 Hz.
and their harmonics. The limit cycle in the map [Fig. 5(e)] is replaced by orbits that are characteristic for quasiperiodic oscillations and cover an area of the plane \((V_1, V_2)\) [Fig. 5(f)]. The correlation function \(r_{V_1V_2}\) becomes periodic with modulation [Fig. 6(b)], and the plot of the correlation dimension \(D_2\) shows the appearance of a two-dimensional process [Fig. 6(j)]. The fluctuations in phase of the neuronal outputs are close to periodic and are larger in amplitude [Fig. 5(j)]. They are in the interval from \(-0.06\) to \(0.04\).

Further increase of \(I_{BSyn}\) to 3.15 nA produces quasiperiodic activity with the frequencies 60.0 and 71.6 Hz and their harmonics [Fig. 6(g)]. The correlation function becomes substantially modulated with a frequency of about 12 Hz that is close to the difference between the two frequencies in quasiperiodic neural outputs [Fig. 5(i)]. The correlation dimension \(D_2\) is equal to 2 [Fig. 6(k)]. A third spectral component appears to be present but is too small to be detected with the correlation function or with calculation of correlation dimension. Fluctuations in the phase of the neural output signals are close to periodic and become even larger (ranged from \(-0.17\) to \(0.09\)).

When \(I_{BSyn}\) increases beyond 3.15 nA, the two-neuron aVLSI system goes into antiphase oscillations. At \(I_{BSyn} = 79\) nA we observed antiphase oscillations with the main frequency at 21.6 Hz [Fig. 6(h)]. These oscillations are substantially non-sinusoidal [Fig. 5(d)] as confirmed by the multiple harmonics in the amplitude spectrum [Fig. 6(b)]. The correlation function is periodic [Fig. 6(d)] and correlation dimension \(D_2\) for this process is equal to 1 [Fig. 6(l)]. The phase of the neuronal outputs is stable around a value of 0.5 [Fig. 5(l)].

A. Mechanism of quasiperiodic oscillation generation

An invariant two-dimensional torus appears through a non-degenerate Neimark–Sacker–Andronov–Hopf torus bifurcation. A stable cycle corresponding to phase-locked oscillations can lose stability when a pair of complex-conjugate multipliers crosses the unit circle. Then, provided that there are no strong resonances, a stable invariant torus is born out of the cycle.\(^{23}\)

Bifurcation of periodic oscillations occurs when symmetric synchronous oscillations are perturbed by the appearance of weak asymmetric mutually inhibitory connection in our two-neuron system. The weakness of interneuron connections is important for the generation of quasiperiodic oscillations that precludes transition to chaotic regime.\(^{19}\)

To elucidate the mechanism of bifurcation of synchronous oscillations into a torus in a two-neuron model system, i.e., how an increase in mutual inhibition through \(I_{BSyn}\) produces instability of periodic oscillations, consider model Eqs. (1)–(4). When \(I_{BSyn} = 0\), both neurons generate identical synchronous periodic oscillations [Fig. 3(a)]. Periodic solution is ensured by a balance between the first three terms in Eqs. (1)–(3), including links between \(V_i\) and \(W_i\) [Eqs. (2) and (4)].

When \(I_{BSyn} > 0\), let consider behavior of a small positive perturbation of \(V_2 (\Delta V_2)\) on the positive half-period of the rising phase of oscillation \((V_2 > V_{th})\). In this case, the last term of Eq. (1) slightly increases as well and causes a small decrease in \(V_1\) because of a decrease in derivative \(dV_1/dt\) comparing to the case \(I_{BSyn} = 0\). A decrease in \(V_1\), in turn, decreases the last term in Eq. (3), increasing further growth of \(V_2\) by increasing derivative \(dV_2/dt\). As result, we have a positive feedback that destabilizes periodic oscillations.

FIG. 5. Experimental data: experimental time series of membrane potentials for two neurons (a)–(d), maps of neuron membrane potentials \(V_i\) versus \(V_j\) (e)–(h), and the phase between the outputs of the two neurons (i)–(l) for aVLSI half-center oscillator system. Synaptic currents \(I_{BSyn}\) are: 2.6 nA (a), (e), (i); 2.7 nA (b), (f), (j); 3.15 nA (c), (g), (k); 79 nA (d), (h), (l).

FIG. 6. Experimental data: experimental cross-correlation functions \(r_{V_1V_2}\) (a)–(d), amplitude spectra for two neurons (e)–(h), and correlation dimensions \(D_2\) (i)–(l) for the aVLSI half-center oscillator system. Synaptic currents \(I_{BSyn}\) correspond to those in Fig. 5. For correlation dimensions only slopes for embedded dimension 21 are shown.
Computer simulations show that this positive feedback leads to bifurcation of periodic oscillations into a torus.

V. DISCUSSION

Torus or quasiperiodic oscillations are oscillatory processes having noncommensurate frequencies. Recent theoretical studies assign an important role to quasiperiodic oscillations in the operation of neuronal networks both living and artificial.\cite{17-20}

How neuronal units may produce such quasiperiodic oscillations is a crucial question for study. Here, by using a physical system that emulates neuronal dynamics (silicon neurons) we show that a ubiquitously observed unit of living neuronal networks, the half-center oscillator, will do the task. Our modeling and experimental studies show that a system of two aVLSI neurons with reciprocal inhibitory connections can robustly generate quasiperiodic oscillations.

Both experimental and model data show the same bifurcation scenario: transitions from synchronous oscillations to quasiperiodic oscillations (torus) to antiphase oscillations. However, the experimental data are less clear than those obtained from the model. For example, experimental synchronous oscillations are less perfectly synchronized, demonstrating deviation from pure straight segment in map $V_1$ versus $V_2$ [compare Figs. 5(e) and 3(e)] and small nonzero component of phase shift $\phi$. The voltage outputs produced by silicon neurons include some noise, also the inherent frequency of each silicon element is slightly different due to limitations in fabrication. These factors disturb phase shift $\phi$ and amplitude spectra.

The silicon neurons and their mathematical model generate oscillations with similar frequencies. Both model and experimental data show quasiperiodic oscillations in a narrow region of $I_{Bsw}$ \cite{34} but the aVLSI two-neuron system shows a narrower region for quasiperiodic oscillations than the mathematical model. The phase shift $\phi$ obtained from the mathematical model have larger oscillation period than that for experimental data [compare Figs. 3(k) and 5(k)]. Nevertheless, quantitative differences in the behavior of the experimental and model systems do not negate that the bifurcation mechanism of synchronous oscillations into a torus is qualitatively the same.

Thus, in this study, we identified a torus bifurcation in a system of symmetrically connected identical oscillators, as a biophysically plausible mechanism for generating quasiperiodic oscillations that are sufficiently robust to persist under physical conditions and potentially be produced in living systems. The quasiperiodic oscillations were observed in spite of noise, fabrication mismatch between the two aVLSI chips, and other influences of the physical world on the silicon neurons dynamics.

A. Tori in biological systems

Since the observations of von Holst of relative coordination in fin movements in fish,\cite{10} similar phenomena have been observed in different biological systems, indicating the existence of quasiperiodic oscillations.\cite{3,21} Such tori indicate a transition from a periodic regime. For example, in quadrupedal locomotion during gait transitions from trot to gallop, one might expect phase oscillations (tori) foreshadowing the transition. Also the observation of such tori may have diagnostic implications for cardiac arrhythmias,\cite{21} and the like.

An important hypothesis describes a potential functional role for quasiperiodic oscillations in the context of synchronization of oscillatory activity of neuronal ensembles. Synchronization is a characteristic of physical, chemical, biological, and social systems.\cite{41-43} From a theoretical perspective,\cite{17-20} oscillatory neuronal networks employing quasiperiodic activity rather than periodic oscillations gain certain functional advantages. Izhikevich demonstrated that the main implications of the theory of weakly coupled periodic oscillators hold for quasiperiodic oscillators.\cite{19} This generalization determines functional advantages for quasiperiodic oscillations. When quasiperiodic activity develops in neural networks, several channels for neuronal communication appear on different noncommensurate frequencies. In this case, neuronal interactions are considerable, if their oscillation frequencies are in resonance or close to resonance,\cite{19,20} and cease otherwise. Thus, quasiperiodic oscillations can increase information flow and improve neural network performance, compared to periodic oscillations with a single frequency. Another consequence is the possibility for a dynamical shaping of effective connectivity in a neuronal network. This property of neural communication on different frequencies can be useful for future development of multitasking neurocomputers and may be realized using two-neuron functional units that generate quasiperiodic oscillations like those studied here.

B. Detection of a torus in a neurophysiological experiment

In physics, quasiperiodic oscillations are commonly observed behavior and a variety of methods have been developed to identify quasiperiodicity. On the other hand, quasiperiodic oscillations are often neglected in experimental neuroscience. Unless special effort is made to identify them, these oscillation may be overlooked in experimental data, being falsely identified as periodic oscillations, a strange attractor, or an artifact of the experiment. This misidentification can imped a rigorous understanding of how nervous systems operate or may cause the discarding of valid experimental data. Here we assembled and tested a set of data analysis techniques that can be used to identify quasiperiodic oscillations in neurophysiological experiments. These methods reliably detect noncommensurate frequencies, when used together in a complimentary fashion. We analyzed phase between voltage outputs, maps of neuronal voltage $V_1$ versus $V_2$, cross-correlation functions, amplitude spectra, and correlation dimensions. Results obtained from all these methods indicate that we have observed the transition from periodic (limit cycle) oscillations into quasiperiodic oscillations in both our mathematical model and experimental data. Except for correlation dimension, these methods are commonly used in neuroscience, and the appearance of quasiperiodic oscillations in neuronal data can be detected by their careful joint application. Using the correlation dimension method in neurophysiological experiments appears to be rather limited be-
cause this method is very sensitive to the length of the time series and noise. However, in the case of low-dimensional quasiperiodic oscillations, the correlation dimension can be useful, because it does not require long time series. It measures dimension of a process, even if it is a nonlinear process. For a nonlinear process, this dimension cannot be clearly determined solely from, for example, the amplitude spectrum, because of nonlinear harmonics [Fig. 6(h) and 6(l)].

C. Where to look for torus

Our study suggests how quasiperiodic oscillations generated by half-center oscillators in the nervous system can be detected experimentally. It will be particularly relevant to systems where partial pharmacological blockade of inhibition causes a transition from anti-phase oscillations into in-phase ones. If such blockade may be controlled in a dose-dependent fashion, we can expect to find a range of doses such that the mutually inhibitory coupling is marginally strong so that quasiperiodic oscillation may exist and be detected. Potentially, these conditions may be met in experiments with spinal locomotory central pattern generators (CPGs) of lamprey.28,29

While in vertebrates preparations experimentalist have to deal with groups of neurons, the CPGs of invertebrates provide an opportunity to study systems of few neurons, and in particular, pairs of mutually inhibitory neurons. For example, Elson et al.43 studied a pair of mutually inhibitory neurons after endogenous synapses were blocked and artificial synapses were introduced with dynamic clamp. Dynamic clamp is a technique which allows one to inject into living neurons currents mimicking synaptic or membrane currents calculated in real time according to a mathematical model. This technique allowed Elson et al.43 to produce transitions from in-phase oscillation into antiphase oscillation and vice versa by carefully varying synaptic time constants. We predict that in this experimental system quasiperiodic oscillations may be revealed as well. By starting with a pair of mutually inhibitory weakly coupled neurons supporting in-phase oscillations and then making synaptic coupling stronger, we expect that before the system makes a transition into anti-phasic oscillations, it will undergo a transition into quasiperiodic oscillations. In our own experiments using leech heartbeat half-center oscillators, we have observed in-phase oscillation during pharmacological blockade of mutual inhibition that suggests that quasiperiodic oscillation might be observed with adjustment of the dose of the pharmacological blocker44 and may be validated with hybrid half-center neural systems that include both silicon and living neurons.44

VI. CONCLUSIONS

Thus, our experimental results and simulations show that a system of two “identical” mutually inhibitory aVLSI neurons demonstrates bifurcation of periodic oscillations into a torus (quasiperiodic oscillations). This bifurcation was identified by the analysis of phases between neuronal outputs, maps of $V_1$ versus $V_2$, amplitude spectra, correlation functions, and correlation dimension. The bifurcation observed in the aVLSI two-neuron system and in its model may occur in central pattern generator neural networks of living organisms. Experimental verification of the scenario, observed in mathematical models and in silicon two-neuron system is proposed.

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